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Green function on the quantum plane

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Abstract. A Green function (which can be called q-analogous of the Hankel function) on the quantum plane $E_q^2 = E_q(2)/U(1)$ is constructed.

1. Introduction

Green functions play important roles in physics. Field theoretical problems involving boundaries, such as Casimir interactions, particle pair productions, for example, all employ Green functions. Therefore, if one is interested in the investigation of some physical effects on non-commutative spaces the construction of Green functions in these media is useful. Motivated by these considerations we believe it is of interest to study Green functions on quantum group spaces which are natural examples of non-commutative geometries.

Previously we have constructed the Green function on the quantum sphere S_q^2 [1]. In this paper we study the same problem for the quantum plane E_q^2 which may be more relevant to physics.

In section 2 we recall the main result [2–6] concerning the quantum group $E_q(2)$ and its homogeneous spaces. In section 3 we construct the Green function on the quantum plane which is the q-deformation of the Hankel function [7].

2. Quantum group $E_q(2)$ and its homogeneous spaces

Let A be the set of linear operators in the Hilbert space $l^2(\mathbb{Z})$ subject to the condition

$$\sum_{j=-\infty}^{\infty} q^{2j}(e_j, F^*Fe_j) < \infty \qquad F \in A. \tag{1}$$

Here $0 < q < 1$ and $\{e_j\}$ is the orthonormal basis in $l^2(\mathbb{Z})$. The explicit form of e_j is

$$e_j = (0, \ldots, 0, 1, 0, \ldots) \tag{2}$$

where either the jth (for $j > 0$) or $(-|j|)$th (for $j < 0$) component is one, all others are zero. Any vector $x = (x_0, x_1, x_{-1}, \ldots, x_n, x_{-n}, \ldots)$ of $l^2(\mathbb{Z})$ has representation

$$x = \sum_{j=-\infty}^{\infty} x_j e_j. \tag{3}$$

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$(\cdot, \cdot)$ in (1) is the scalar product in $l^2(\mathbb{Z})$:

$$(x, y) = \sum_{j=-\infty}^{\infty} \overline{x_j} y_j. \tag{4}$$

A is the Hilbert space with the scalar product

$$(F, G)_A = (1 - q^2) \sum_{j=-\infty}^{\infty} q^{2j} (e_j, F^* G e_j) \qquad F, G \in A. \tag{5}$$

Let us introduce the linear operators acting in $l^2(\mathbb{Z})$:

$$z e_j = \mathrm{e}^{\mathrm{i}\psi} q^j e_j \qquad \upsilon e_j = \mathrm{e}^{\mathrm{i}\phi} \mathrm{e}_{j+1} \tag{6}$$

where ψ and ϕ are the classical phase variables. z is normal and υ is the unitary operator in $l^2(\mathbb{Z})$. It is easy to show that they satisfy the relations

$$z\upsilon = q\upsilon z \qquad z^*\upsilon = q\upsilon z^* \qquad zz^* = z^* z. \tag{7}$$

Any element $F \in A$ can be represented as

$$F = \sum_{j=-\infty}^{\infty} f_j(z, z^*) \upsilon^j \tag{8}$$

by suitable choice of the functions f_j.

The linear operators Z and V given by

$$Z = z \otimes \upsilon^{-1} + \upsilon \otimes z \qquad V = \upsilon \otimes \upsilon \tag{9}$$

are normal and unitary in $l^2(\mathbb{Z} \times \mathbb{Z})$. They satisfy the relations

$$ZV = qVZ \qquad Z^*V = qVZ^* \qquad ZZ^* = Z^*Z. \tag{10}$$

Note that the operators Z and V have the same properties as z and υ. Therefore, there exits the linear map [2]

$$\Delta : A \to A \otimes_A A \tag{11}$$

defined as

$$\Delta(f(z, z^*)\upsilon^j) = f(Z, Z^*) V^j. \tag{12}$$

Here $\otimes_A$ is the completed tensor product $\otimes$ with respect to the scalar product

$$(F_1 \otimes F_2, F_3 \otimes F_4)_A = (F_1, F_3)_A (F_2, F_4)_A \qquad F_n \in A \tag{13}$$

in $A \otimes A$. A is the space of square integrable functions on the quantum group $E_q(2)$ and Δ is the quantum analogue of the group multiplication.

The one-parameter groups $\{\sigma_1\}$ and $\{\sigma_2\}$ of the automorphism of A given by

$$\sigma_1(\upsilon) = \mathrm{e}^{-\mathrm{i}t} \upsilon \qquad \sigma_1(z) = \mathrm{e}^{\mathrm{i}t} z \tag{14}$$

and

$$\sigma_2(\upsilon) = \upsilon \qquad \sigma_2(z) = \mathrm{e}^{\mathrm{i}t} z \tag{15}$$

with $t \in \mathbb{R}$ are isomorphic to $U(1)$. The subspaces

$$B = \{F \in A : \sigma_1(F) = F, \text{ for all } \quad t \in \mathbb{R}\} \tag{16}$$

and

$$H = \{F \in B : \sigma_2(F) = F, \text{ for all } \quad t \in \mathbb{R}\} \tag{17}$$

are the spaces of square integrable functions on the quantum plane E_q^2 and the two-sided coset space $U(1)\backslash E_q(2)/U(1)$. Any element of H is the function of $\rho = zz^*$. Note that the scalar product (5) on H becomes a q-integration

$$(f(\rho), g(\rho))_A = (1-q^2) \sum_{j=-\infty}^{\infty} q^{2j} \overline{f(q^{2j})} g(q^{2j}) = \int_0^\infty \overline{f(\rho)} g(\rho)\, \mathrm{d}_{q^2}\rho. \tag{18}$$

Let $U_q(e(2))$ be the $*$-algebra generated by p and $\kappa^{\pm 1}$ such that

$$p^* p = q^2 pp^* \qquad \kappa^* = \kappa \qquad \kappa p = q^2 p\kappa. \tag{19}$$

The representation $\mathcal{L}$ of $U_q(e(2))$ in $D \subset A$ is given by

$$\mathcal{L}(p) f(z, z^*)\upsilon^j = \mathrm{i} q^{j+1} D_+^z f(z, z^*)\upsilon^{j+1} \tag{20}$$

$$\mathcal{L}(p^*) f(z, z^*)\upsilon^j = \mathrm{i} q^{j-1} D_-^{z^*} f(z, z^*)\upsilon^{j-1} \tag{21}$$

$$\mathcal{L}(\kappa) f(z, z^*)\upsilon^j = q^j f(q^{-1}z, qz^*)\upsilon^j \tag{22}$$

where

$$D_\pm^x f(x) = \frac{f(x) - f(q^{\pm 2}x)}{(1-q^{\pm 2})x}. \tag{23}$$

Let us define the common invariant domain D for the algebra $U_q(e(2))$ which is dense in A. Using the spectral decomposition of the operator functions $f_j(z, z^*)$ we consider the vector function $\{f_j(\zeta, \overline{\zeta})\}_{j=-\infty}^{\infty}$, $\zeta \in \Omega$, where

$$\Omega = \{\zeta \in \mathbb{C} : |\zeta| = \{q^j, j \in \mathbb{Z}\} \cup \{0\}\} \tag{24}$$

in place of $F \in A$ given by (8). The norm (1) then reads

$$\|F\|^2 = \sum_{j=-\infty}^{\infty} q^{-2j} \int_\Omega \mathrm{d}\sigma(\zeta, \overline{\zeta}) |f_j(\zeta, \overline{\zeta})|^2 \tag{25}$$

where the integration measure σ is given by

$$\mathrm{d}\sigma(\zeta, \overline{\zeta}) = \begin{cases} q^{2j} & \text{if} \quad |\zeta| = q^j \qquad j \in \mathbb{Z} \\ 0 & \text{if} \quad |\zeta| \neq q^j \qquad j \in \mathbb{Z}. \end{cases} \tag{26}$$

D is then defined as:

(1) functions $f_j(\zeta, \overline{\zeta})$ are infinitely differentiable on Ω;
(2) functions $f_j(\zeta, \overline{\zeta})$ have finite support on Ω;
(3) only the finite number of the components of the vector $\{f_j(\zeta, \overline{\zeta})\}_{j=-\infty}^{\infty}$ are non-zero.

For the Casimir element $C = -q^{-1}\kappa^{-1}pp^*$ we have

$$\mathcal{L}(C) f(z, z^*)\upsilon^j = q^j D_-^{z^*} D_+^z f(qz, q^{-1}z^*)\upsilon^j. \tag{27}$$

The restriction $\Box$ of $\mathcal{L}(C)$ on $H_0 = H \cap D$ is

$$\Box = D_-^\rho \rho D_+^\rho \tag{28}$$

which we call the radial part of $\mathcal{L}(C)$.

3. Green function on the quantum plane

3.1. Green function on $U(1)\backslash E_q(2)/U(1)$

The Green function $\mathcal{G}^p(\rho)$ on the two-sided coset space is defined as

$$(\Box + p)\mathcal{G}^p(\rho) = \delta(\rho) \tag{29}$$

where δ is the delta function which is defined with respect to the scalar product (18) as

$$(\delta, f)_A = f(0) \tag{30}$$

for any $f \in H_0$. Equation (29) is understood as

$$(\mathcal{G}_p, f)_A = \lim_{\epsilon\to 0}(\delta(\rho) \qquad \frac{1}{\Box + p + \mathrm{i}\epsilon} f(\rho))_A. \tag{31}$$

For $\rho \neq 0$ the equation (29) is solved by

$$\mathcal{J}(\sqrt{p\rho}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{([k]!)^2}(p\rho)^k \tag{32}$$

and

$$\mathcal{N}\left(\sqrt{p\rho}\right) = \frac{q - q^{-1}}{2q\log(q)}\mathcal{J}(\sqrt{p\rho})(\log(p\rho) + 2C_q) - \frac{1}{q}\sum_{k=1}^{\infty}\frac{(-1)^k}{([k]!)^2}(p\rho)^k \sum_{m=1}^{k}\frac{q^m + q^{-m}}{[m]} \tag{33}$$

where

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}} \qquad [m]! = [1][2]\ldots[m]. \tag{34}$$

The Hahn–Exton q-Bessel function $\mathcal{J}$ is regular at $\rho = 0$. It is the zonal spherical function of the unitary irreducible representations of $E_q(2)$. $\mathcal{N}$ can be called the q-Neuman function which is indeed reduced to the usual Neuman function in the $q \to 1$ limit. Here C_q is some constant, which in the $q \to 1$ limit should become the Euler constant [8].

The Green function on $U(1)\backslash E_q(2)/U(1)$ is then

$$\mathcal{G}_p(\rho) = \mathcal{J}\left(\sqrt{p\rho}\right) + \mathrm{i}\mathcal{N}\left(\sqrt{p\rho}\right) \tag{35}$$

which in the classical limit becomes the Hankel function. Using the Fourier–Bessel integral [9]

$$\int_0^{\infty} \mathrm{d}_{q^2}\rho\, \mathcal{J}\left(q^n\sqrt{\rho}\right)\mathcal{J}(q^m\sqrt{\rho}) = \frac{q^{2m+2}}{1 - q^2}\delta_{mn} \tag{36}$$

we arrive at the following representation for the Green function:

$$\mathcal{G}_p(\rho) = \lim_{\epsilon\to 0} q^{-2}\int_0^{\infty} \mathrm{d}_{q^2}\lambda \frac{\mathcal{J}(\sqrt{\lambda\rho})}{p - \lambda + \mathrm{i}\epsilon} \tag{37}$$

from which one can derive the constant C_q.

To prove that $\mathcal{G}$ solves (29) we first have to show that

$$\Box \log\rho = \frac{2q\log(q)}{q - q^{-1}}\delta(\rho). \tag{38}$$

For $\rho \neq 0$ we have

$$\Box \log\rho = 0. \tag{39}$$

Since the operator $\Box$ is symmetric in H we have

$$(\Box \log\rho, f)_A = (\log\rho, \Box f)_A = \frac{2q\log(q)}{q-q^{-1}} \lim_{n\to\infty} \sum_{j=-\infty}^{n} j[2f(q^{2j}) - f(q^{2(j+1)}) - f(q^{2(j-1)})] = \frac{2q\log(q)}{q-q^{-1}} \lim_{n\to\infty} [f(q^{2n}) - n(f(q^{2n+2}) - f(q^{2n}))]. \tag{40}$$

We then employ the q-Taylor expansion at the neighbourhood of $\rho = 0$:

$$f(q^2\rho) - f(\rho) \sim D_+^\rho f(0)(q^2-1)\rho. \tag{41}$$

For $n \gg 1$ we get

$$n(f(q^{2n+2}) - f(q^{2n})) \sim nD_+^\rho f(0)(q^2-1)q^{2n}. \tag{42}$$

Since nq^{2n} vanishes as $n \to \infty$, we arrive at

$$(\Box \log\rho, f)_A = \frac{2q\log(q)}{q-q^{-1}} \lim_{n\to\infty} f(q^{2n}) = \frac{2q\log(q)}{q-q^{-1}} f(0). \tag{43}$$

In a similar fashion one can show that

$$((\Box + p)\mathcal{G}^p(\rho), f)_A = f(0). \tag{44}$$

3.2. Green function on E_q^2

We obtain the Green function $\mathcal{G}^p(R)$ on the quantum plane E_q^2 from the one $\mathcal{G}^p(\rho)$ on the two-sided coset space by the group multiplication [1]:

$$\mathcal{G}^p(R) = \Delta\mathcal{G}^p(\rho). \tag{45}$$

Here

$$R = \Delta(\rho) = \rho\otimes 1 + 1\otimes\rho + \upsilon z^*\otimes z\upsilon + z\upsilon^*\otimes\upsilon^* z^* \tag{46}$$

is the self-adjoint operator in $l^2(\mathbb{Z}\times\mathbb{Z})$ and

$$Re_{ts} = q^{2t}e_{ts} \tag{47}$$

where the eigenfunctions e_{ts} are given by [3]

$$e_{ts} = \sum_{j=-\infty}^{\infty} (-1)^j q^{t-j} \mathcal{J}_s(q^{t-j}) e_{s+j}\otimes e_j. \tag{48}$$

They satisfy the orthogonality condition

$$(e_{ts}, e_{ij}) = \delta_{ti}\delta_{sj}. \tag{49}$$

We also have

$$e_{s+j}\otimes e_j = \sum_{t=-\infty}^{\infty} (-1)^j q^{t-j} \mathcal{J}_s(q^{t-j}) e_{ts}. \tag{50}$$

Therefore, the basis elements e_{ts}; $t, s \in (-\infty, \infty)$ form the complete set in $l^2(\mathbb{Z}\times\mathbb{Z})$. The Green function on the quantum plane is the linear operator in this space defined as

$$\mathcal{G}^p(R)e_{ts} = \mathcal{G}^p(q^{2t})e_{ts}. \tag{51}$$

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